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1993 J. Phys. A: Math. Gen. 26 2179

(http://iopscience.iop.org/0305-4470/26/9/014)

J. Phys. A: Math. Gen. **26** (1993) 2179–2197. Printed in the UK

S-function series revisited*

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Received 17 June 1992, in final form 14 December 1992

Abstract. A systematic and readily programmable computational method is given to extract the S-function content of any arbitrary symmetric polynomial function. Applications to the derivation of new series and the outer product or skew division of an S-function by a series are shown.

1. Introduction

Schur functions or S-functions are symmetric polynomials whose importance stems from the fact that they play a major role in the character theory of compact (Littlewood 1950, Wybourne 1970, Macdonald 1979) as well as non-compact Lie groups (Rowe *et al* 1985, King and Wybourne 1985). Operations on S-functions have been studied extensively, several S-functions series are now well established and various methods have been devised to convert generating functions to their S-function content and *vice versa* (Littlewood 1950, Bender and Knuth 1972, McConnell and Newell 1973, Burge 1974, King 1975, Macdonald 1979, King *et al* 1981, Black *et al* 1983, Josefiak and Weyman 1985, Yang and Wybourne 1986, Lascoux and Pragacz 1988).

However, the importance of S-function series is not limited to the so-called 'classical' ones as Yang and Wybourne (1986) so rightfully pointed out in their systematic study of series. For example in the framework of the symplectic shell model (and their submodels) nuclear configurations are labelled by $\langle\{\lambda\}\rangle$ (Rowe *et al* 1985, King and Wybourne 1985), the infinite-dimensional (holomorphic discrete series) irreps of the symplectic group $\mathrm{Sp}(2N,\mathbb{R})$ with $N=3$, where $\{\lambda\}$ is a standard S-function in not more than three parts. Since under the restriction of $\mathrm{Sp}(2N,\mathbb{R})$ to its subgroup $\mathrm{U}(N)$ the symplectic irrep decomposes as

$$\langle\{\lambda\}\rangle \downarrow \{\lambda\}\cdot D \tag{1.1}$$

and, inversely, one has

$$\{\mu\} \uparrow \langle\{\mu\}\cdot C\rangle \tag{1.2}$$

the product of two symplectic irreps is given by

$$\begin{aligned}\langle\{\lambda\}\rangle\langle\{\nu\}\rangle &= \sum\langle\{\chi\}\rangle \\ \sum\langle\{\chi\}\rangle &= \langle\{\lambda\}\{\nu\}\cdot D\rangle\end{aligned} \tag{1.3}$$

* Work supported by a grant from NSERC.

as can be seen by considering

$$\mathrm{Sp}(2N,\mathbb{R}) \downarrow \mathrm{U}(N) \uparrow \mathrm{Sp}(2N,\mathbb{R})$$

$$\langle\{\lambda\}\rangle\langle\{\nu\}\rangle \downarrow \{\lambda\}\cdot D\{\nu\}\cdot D \uparrow \langle\{\lambda\}\{\nu\}\cdot D\cdot D\cdot C\rangle. \tag{1.4}$$

The exploitation of (1.3) then involves the evaluation of the S-function content of the infinite series $\{\lambda\}\{\nu\}\cdot D$.

On the other hand, it was shown in an earlier paper (Carvalho 1990) that the symplectic irreps that span the configuration space of a nucleus of mass number A can be found by evaluating the product of A copies of the infinite series of S-functions M followed by the product with the series C , i.e.

$$\sum\langle\{\lambda\}\rangle = \langle\underbrace{M\cdot M\cdot M\cdots M}_{A\ \text{times}}\cdot C\rangle. \tag{1.5}$$

Now, if the S-function content of either compound series (1.3) or (1.5) is obtained from the known S-function content of the classical series D or M and C, the resulting expansions appear ordered according to the increasing weight of the S-functions or, equivalently, the symplectic irreps appear ordered according to the increasing spherical harmonic oscillator energy content. However, neither do all irreps correspond to allowed nuclear configurations (due to the Pauli principle) nor are the configurations of lower spherical harmonic oscillator energy necessarily the most important ones.

For $A = 16$, for example, irreps $\langle\{\lambda\}\rangle$ for which the weight of the corresponding S-functions $\{\lambda\}$ is less than 12 do not correspond to possible configurations of ^{16}O and, even for a light nucleus such as ^{16}O, a configuration labelled by an S-function of weight 16 is more relevant to the description of its low-lying spectrum than one corresponding to an S-function of weight 14.

Obviously, as the mass number A increases the minimum weight physically allowed increases and the relevant symplectic irreps are found at quite high spherical harmonic oscillator energy levels compared to the minimum. It is then desirable to have an economical method that produces those S-functions in the expansion that are of physical interest without having to generate all of them.

It is with the above mentioned problem in mind and the fact that other algebraic models may encounter similar problems involving different series that the objective of this paper is to give a systematic procedure for the computation of the S-function content of any finite or infinite symmetric polynomial function.

The organization of the paper is as follows. Definitions and relations between polynomials and S-functions are reviewed in section 2. In section 3, McConnell's and Newell's method is applied to a general symmetric polynomial. The S-function content of three general types of series is derived in section 4 and, in section 5, an algorithm is presented to calculate the multiplicity of a given S-function in the expansions. Finally, in section 6, some examples of the applicability of the method are shown.

All illustratives examples throughout the text are kept simple so that the reader may check the results quickly by hand.

2. Polynomials and S-functions

A general S-function (or Schur-function), labelled by a set of l integers, called parts, $\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$, is a symmetric polynomial function of a set of $p \geqslant l$ indeterminates

$\alpha_1, \alpha_2, \ldots, \alpha_p$ which can be given conveniently by the bideterminantal formula (McConnell and Newell 1973),

$$\det(\alpha_t^{\lambda_s+p-s}) / \det(\alpha_t^{p-s}) \tag{2.1}$$

where $t, s = 1, \ldots, p$ denote the rows and columns respectively of the $p \times p$ determinants. The integer $l \leqslant p$ specifies the 'length' of the S-function, (i.e. the number of its non-vanishing parts when in standard form) and $w = \lambda_1 + \lambda_2 + \cdots + \lambda_l$ denotes its 'weight'.

For example, the sum of the monomial symmetric functions (Macdonald 1979)

$$h_n = \sum_{|\nu|=n} m_\nu = \sum \alpha_{i_1}^n + \sum \alpha_{i_1}^{n-1}\alpha_{i_2} + \sum \alpha_{i_1}^{n-2}\alpha_{i_2}^2 + \sum \alpha_{i_1}^{n-2}\alpha_{i_2}\alpha_{i_3} + \cdots + \begin{cases} \sum \alpha_{i_1}^{n-p+1}\alpha_{i_2}\cdots\alpha_{i_p} & \text{if } n > p \\ \sum \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n} & \text{if } n \leqslant p \end{cases} \tag{2.2}$$

where each sum is taken over only those values of $i_1, i_2, \ldots$ leading to distinct terms, corresponds to the complete symmetric S-function $\{n\}$, in one part only, which in accordance with definition (2.1) can be written as

$$\det \begin{pmatrix} \alpha_1^{n+p-1} & \alpha_1^{p-2} & \cdots & \alpha_1^0 \\ \alpha_2^{n+p-1} & \alpha_2^{p-2} & \cdots & \alpha_2^0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_p^{n+p-1} & \alpha_2^{p-2} & \cdots & \alpha_p^0 \end{pmatrix} / \Delta(\alpha) \tag{2.3}$$

where $\Delta(\alpha)$ is the Vandermonde determinant

$$\Delta(\alpha) = \det(\alpha_t^{p-s}) = \prod_{i<j}(\alpha_i - \alpha_j). \tag{2.4}$$

On the other hand, the elementary symmetric functions (Macdonald 1979)

$$e_n = \sum \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n} \qquad \text{with} \qquad n \leqslant p \tag{2.5}$$

can be expressed in determinantal form as

$$\det \begin{pmatrix} \alpha_1^{1+p-1} & \alpha_1^{1+p-2} & \cdots & \alpha_1^{1+p-n} & \cdots & \alpha_1^0 \\ \alpha_2^{1+p-1} & \alpha_2^{1+p-2} & \cdots & \alpha_2^{1+p-n} & \cdots & \alpha_2^0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \alpha_p^{1+p-1} & \alpha_p^{1+p-2} & \cdots & \alpha_p^{1+p-n} & \cdots & \alpha_p^0 \end{pmatrix} / \Delta(\alpha) \tag{2.6}$$

and corresponds to the S-function $\{1^n\}$ (cf (2.1)).

An S-function $\{\lambda\}$ (s_λ in Macdonald's notation) is said to be 'standard' if its parts satisfy the condition $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p \geqslant 0$.

Non-standard S-functions can be converted to standard ones by successive applications of the modification rule (Littlewood 1950)

$$\{\lambda_1, \lambda_2, \ldots, \lambda_{i-k}, \ldots, \lambda_i, \ldots, \lambda_p\} = -\{\lambda_1, \lambda_2, \ldots, \lambda_i - k, \ldots, \lambda_{i-k} + k, \ldots, \lambda_p\} \tag{2.7}$$

derived from the rule for interchanging two columns in a determinant and the definition (2.1). On the other hand, in conformity with the fact that a determinant is zero if two of its columns are identical, a non-standard S-function vanishes if two of its parts are related by (cf again (2.1))

$$\lambda_{i-k} = \lambda_i - k. \tag{2.8}$$

Any standard S-function in l parts has $l! - 1$ equivalent non-standard counterparts. For example, the standard S-function in three parts, $\{\lambda_1, \lambda_2, \lambda_3\}$, is equivalent to the five non-standard S-functions indicated in the following diagram:

$$\begin{array}{ccccc} & & \{\lambda_1, \lambda_2, \lambda_3\} & & \\ & \swarrow & & \searrow & \\ -\{\lambda_2 - 1, \lambda_1 + 1, \lambda_3\} & & & & -\{\lambda_1, \lambda_3 - 1, \lambda_2 + 1\} \\ \downarrow & & & & \downarrow \\ +\{\lambda_2 - 1, \lambda_3 - 1, \lambda_1 + 2\} & & & & +\{\lambda_3 - 2, \lambda_1 + 1, \lambda_2 + 1\} \\ & \searrow & & \swarrow & \\ & & -\{\lambda_3 - 2, \lambda_2, \lambda_1 + 2\} & & \end{array} \tag{2.9}$$

where the + or − signs are dictated by relation (2.7). If, for the particular values assumed by λ_i, either the ith part, μ_i, of a non-standard S-function satisfies $\mu_i < i - l$ or any two of its parts are related by (2.8), then this non-standard S-function vanishes identically.

3. McConnell's and Newell's method

The method of determining the S-function content of a series by means of an intermediate determinantal form of the generating function was first used, to great effect, by McConnell and Newell (1973). The technique is reviewed here as it is applied to obtain the S-function expansion of symmetric polynomials of the type

$$\sum \alpha_1^{q_1} \alpha_2^{q_2} \dots \alpha_p^{q_p} \tag{3.1}$$

where $q_1, q_2, \dots, q_p$ are fixed non-negative integers and the sum includes all $p!$ permutations of $q_1, q_2, \dots, q_p$. Note that the number of indeterminates in which the polynomial function is expressed determines the maximum length of the S-functions appearing in the expansion.

In the following, and for the sake of simplicity of presentation, the number of indeterminates is restricted to $p = 3$. Generalization of the results to a larger number of indeterminates is, however, straightforward. Let us then consider the S-function content of $\sum \alpha_1^{q_1} \alpha_2^{q_2} \alpha_3^{q_3}$.

The first step in implementing McConnell's and Newell's method is to multiply the given polynomial by $\Delta(\alpha)$ (cf equation (2.4))

$$\alpha_1^{q_1}\alpha_2^{q_2}\alpha_3^{q_3} \times \Delta(\alpha) + \alpha_2^{q_1}\alpha_1^{q_2}\alpha_3^{q_3} \times \Delta(\alpha) + \alpha_1^{q_1}\alpha_3^{q_2}\alpha_2^{q_3} \times \Delta(\alpha)$$
$$+ \alpha_3^{q_1}\alpha_1^{q_2}\alpha_2^{q_3} \times \Delta(\alpha) + \alpha_2^{q_1}\alpha_3^{q_2}\alpha_1^{q_3} \times \Delta(\alpha) + \alpha_3^{q_1}\alpha_2^{q_2}\alpha_1^{q_3} \times \Delta(\alpha) \tag{3.2a}$$

which produces the following six determinants

$$\det\begin{pmatrix}\alpha_1^{q_1+2} & \alpha_1^{q_1+1} & \alpha_1^{q_1+0}\\ \alpha_2^{q_2+2} & \alpha_2^{q_2+1} & \alpha_2^{q_2+0}\\ \alpha_3^{q_3+2} & \alpha_3^{q_3+1} & \alpha_3^{q_3+0}\end{pmatrix}+\det\begin{pmatrix}\alpha_1^{q_2+2} & \alpha_1^{q_2+1} & \alpha_1^{q_2+0}\\ \alpha_2^{q_1+2} & \alpha_2^{q_1+1} & \alpha_2^{q_1+0}\\ \alpha_3^{q_3+2} & \alpha_3^{q_3+1} & \alpha_3^{q_3+0}\end{pmatrix}$$
$$+\det\begin{pmatrix}\alpha_1^{q_1+2} & \alpha_1^{q_1+1} & \alpha_1^{q_1+0}\\ \alpha_2^{q_3+2} & \alpha_2^{q_3+1} & \alpha_2^{q_3+0}\\ \alpha_3^{q_2+2} & \alpha_3^{q_2+1} & \alpha_3^{q_2+0}\end{pmatrix}+\det\begin{pmatrix}\alpha_1^{q_2+2} & \alpha_1^{q_2+1} & \alpha_1^{q_2+0}\\ \alpha_2^{q_3+2} & \alpha_2^{q_3+1} & \alpha_2^{q_3+0}\\ \alpha_3^{q_1+2} & \alpha_3^{q_1+1} & \alpha_3^{q_1+0}\end{pmatrix}$$
$$+\det\begin{pmatrix}\alpha_1^{q_3+2} & \alpha_1^{q_3+1} & \alpha_1^{q_3+0}\\ \alpha_2^{q_1+2} & \alpha_2^{q_1+1} & \alpha_2^{q_1+0}\\ \alpha_3^{q_2+2} & \alpha_3^{q_2+1} & \alpha_3^{q_2+0}\end{pmatrix}+\det\begin{pmatrix}\alpha_1^{q_3+2} & \alpha_1^{q_3+1} & \alpha_1^{q_3+0}\\ \alpha_2^{q_2+2} & \alpha_2^{q_2+1} & \alpha_2^{q_2+0}\\ \alpha_3^{q_1+2} & \alpha_3^{q_1+1} & \alpha_3^{q_1+0}\end{pmatrix}. \qquad (3.2b)$$

The next step is to regroup the monomial terms in such way that they can be arranged into six determinants each of the type given in the numerator of definition (2.1):

$$\det\begin{pmatrix}\alpha_1^{q_1+2} & \alpha_1^{q_2+1} & \alpha_1^{q_3+0}\\ \alpha_2^{q_1+2} & \alpha_2^{q_2+1} & \alpha_2^{q_3+0}\\ \alpha_3^{q_1+2} & \alpha_3^{q_2+1} & \alpha_3^{q_3+0}\end{pmatrix}+\det\begin{pmatrix}\alpha_1^{q_2+2} & \alpha_1^{q_1+1} & \alpha_1^{q_3+0}\\ \alpha_2^{q_2+2} & \alpha_2^{q_1+1} & \alpha_2^{q_3+0}\\ \alpha_3^{q_2+2} & \alpha_3^{q_1+1} & \alpha_3^{q_3+0}\end{pmatrix}$$
$$+\det\begin{pmatrix}\alpha_1^{q_1+2} & \alpha_1^{q_3+1} & \alpha_1^{q_2+0}\\ \alpha_2^{q_1+2} & \alpha_2^{q_3+1} & \alpha_2^{q_2+0}\\ \alpha_3^{q_1+2} & \alpha_3^{q_3+1} & \alpha_3^{q_2+0}\end{pmatrix}+\det\begin{pmatrix}\alpha_1^{q_3+2} & \alpha_1^{q_1+1} & \alpha_1^{q_2+0}\\ \alpha_2^{q_3+2} & \alpha_2^{q_1+1} & \alpha_2^{q_2+0}\\ \alpha_3^{q_3+2} & \alpha_3^{q_1+1} & \alpha_3^{q_2+0}\end{pmatrix}$$
$$+\det\begin{pmatrix}\alpha_1^{q_2+2} & \alpha_1^{q_3+1} & \alpha_1^{q_1+0}\\ \alpha_2^{q_2+2} & \alpha_2^{q_3+1} & \alpha_2^{q_1+0}\\ \alpha_3^{q_2+2} & \alpha_3^{q_3+1} & \alpha_3^{q_1+0}\end{pmatrix}+\det\begin{pmatrix}\alpha_1^{q_3+2} & \alpha_1^{q_2+1} & \alpha_1^{q_1+0}\\ \alpha_2^{q_3+2} & \alpha_2^{q_2+1} & \alpha_2^{q_1+0}\\ \alpha_3^{q_3+2} & \alpha_3^{q_2+1} & \alpha_3^{q_1+0}\end{pmatrix} \qquad (3.3)$$

and now dividing (3.3) by $\Delta(\alpha)$ to recover the original polynomial, we get, according to equation (2.1), the following set of S-functions

$$\sum \alpha_1^{q_1}\alpha_2^{q_2}\alpha_3^{q_3} = \{q_1q_2q_3\}+\{q_2q_1q_3\}+\{q_1q_3q_2\}+\{q_3q_1q_2\}+\{q_2q_3q_1\}+\{q_3q_2q_1\}. \qquad (3.4)$$

Note that there are as many S-functions in the expansion (3.4), as monomial terms in the original polynomial. However, each S-function in (3.4) is built from contributions from all six monomial terms and the absence of one of them (unless identical to another existing term) makes the construction of the whole S-function expansion impossible. Note also that if $q_1 \neq q_2 \neq q_3$ only one of the S-functions is standard. Although for the objective of this paper it is not important to have the S-functions in standard form it is obvious that application of the modification rules to (3.4) yields a result analogous to that of Littlewood (1950), namely

$$\sum \alpha_1^{q_1}\alpha_2^{q_2}\alpha_3^{q_3} = \sum C^{(\lambda)}\{\lambda\}$$

where $\{\lambda\}$ are standard S-functions, in three parts, of weight $q_1+q_2+q_3$ and $C^{(\lambda)}$ their multiplicity.

4. Classical S-function series and their derivatives

What is commonly called a 'series of S-functions' (Littlewood 1950) is a set of S-functions which have the same origin, the so-called 'generating function'. The basic difference between a series and, for example, expansion (3.4) is the fact that the S-functions that constitute the series have increasing weights and $\{0\} = 1$ is always the first term. Some generating functions produce infinite series (i.e. there is no limit for the maximum weight) regardless of the number of indeterminates in the polynomial function; others yield a finite number of S-functions, which is dependent on the number of arguments of the generating function. Three basic types of generating functions will be discussed in the following.

4.1. Finite series

4.1.1. S-function series produced by the generating function.

$$\prod_i^p (1-\alpha_i^r)^k \qquad \text{with } r, k > 0. \tag{4.1}$$

For $k = 1$ and $r = 1$ the resulting series has been identified in the literature (Yang and Wybourne 1986) as the series L; for $k = 1$ and $r = 2$ it is the series V.

Defining the 'symmetric power sum' functions (Wybourne 1970, Macdonald 1979), p_r, of the indeterminates α_i, by

$$p_r = \sum_i \alpha_i^r$$

and making use of the property that for any polynomial $P(\alpha)$

$$P(\alpha) \otimes p_r = P(\alpha^r)$$

where $\otimes$ denotes symmetrized product or 'plethysm', one concludes that (4.1) is then the generating function of the series $[L \otimes p_r]^k$.

The S-function content of this series can be found by expanding the generating function (4.1) as a sum of symmetric polynomials of the type (3.1), as follows

$$\begin{aligned}\prod_i^p (1-\alpha_i^r)^k &= \prod_i^p \sum_{l=0}^k \binom{k}{l} (-1)^l \alpha_i^{rl} \\ &= \sum_{l_1=0}^k \sum_{l_2=0}^k \cdots \sum_{l_p=0}^k a(k, l_1, l_2, \ldots, l_p)\, \alpha_1^{rl_1} \alpha_2^{rl_2} \ldots \alpha_p^{rl_p}\end{aligned} \tag{4.2}$$

where

$$a(k, l_1, l_2, \ldots, l_p) = (-1)^{l_1+l_2+\cdots+l_p} \binom{k}{l_1} \binom{k}{l_2} \cdots \binom{k}{l_p} \tag{4.3a}$$

and

$$\binom{k}{l_i} = \frac{k!}{l_i!(k-l_i)!} \qquad \text{with } i = 1, 2, \ldots, p. \tag{4.3b}$$

Since l_1, l_2, ..., l_p take, independently, all integer values from zero to k and the coefficients $a(k, l_1, l_2, \ldots, l_p)$ are invariant under any permutation of a particular sequence of values $(l_1, l_2, \ldots, l_p)$, expansion (4.2) can be written as

$$\sum_{l_1, l_2, \ldots, l_p} a(k, l_1, l_2, \ldots, l_p) \sum \alpha_1^{rl_1} \alpha_2^{rl_2} \ldots \alpha_p^{rl_p}$$

where, according to (3.4),

$$\sum \alpha_1^{rl_1} \alpha_2^{rl_2} \ldots \alpha_p^{rl_p}$$

produces $p!$ S-functions labelled by all permutations of $rl_1, rl_2, \ldots, rl_p$. Therefore the S-function content of (4.1) is

$$\sum_{l_1=0}^{k} \sum_{l_2=0}^{k} \cdots \sum_{l_p=0}^{k} a(k, l_1, l_2, \ldots, l_p) \{rl_1, rl_2, \ldots rl_p\}. \tag{4.4}$$

For example, for $p = 3$, $r = 2$ and $k = 3$ we have the following series:

$$V^3 = \prod_i^3 (1 - \alpha_i^2)^3 = \sum_{l_1=0}^{3} \sum_{l_2=0}^{3} \sum_{l_3=0}^{3} (-1)^{l_1+l_2+l_3} \binom{3}{l_1} \binom{3}{l_2} \binom{3}{l_3} \{2l_1, 2l_2, 2l_3\} \tag{4.5a}$$

which can be quickly worked out to give, after conversion of the non-standard S-functions,

$$\begin{aligned} V^3 = \{0\} &- 3\{2\} + 3\{11\} + 3\{4\} - 3\{31\} + 9\{22\} - 6\{211\} \\ &- \{6\} + \{51\} - 9\{42\} + 8\{411\} + 9\{33\} - 18\{222\} \\ &+ 3\{62\} - 3\{611\} - 3\{53\} + 9\{44\} - 6\{431\} + 24\{422\} - 18\{332\} \\ &- 3\{64\} + 3\{631\} - 9\{622\} + 3\{55\} + 6\{532\} - 24\{442\} + 18\{433\} \\ &+ \{66\} - \{651\} + 9\{642\} - 9\{633\} - 8\{552\} + 18\{444\} \\ &- 3\{662\} + 3\{653\} - 9\{644\} + 6\{554\} + 3\{664\} - 3\{655\} - \{666\}. \end{aligned} \tag{4.5b}$$

4.1.2. *S-function series produced by the generating function*

$$\prod_{i<j} (1 - (\alpha_i \alpha_j)^r)^k \qquad \textit{with } r, k > 0. \tag{4.6}$$

For $r = k = 1$, (4.6) is the generating function of the classical series A. Thus

$$\prod_{i<j} (1 - (\alpha_i \alpha_j)^r)^k = [A \otimes p_r]^k. \tag{4.7}$$

Expanding the polynomial function yields

$$\begin{aligned} \prod_{i<j} (1 - (\alpha_i \alpha_j)^r)^k &= \prod_{i<j} \sum_{l=0}^{k} (-1)^l \binom{k}{l} (\alpha_i \alpha_j)^{rl} \\ &= \sum_{l_{12} l_{13} \ldots l_{1p}}^{k} \sum_{l_{23} l_{24} \ldots l_{2p}}^{k} \cdots \sum_{l_{p-1p}}^{k} a(k, l_{12}, l_{13}, \ldots, l_{p-1p}) \\ &\quad \times \alpha_1^{r(l_{12}+l_{13}+\cdots+l_{1p})} \alpha_2^{r(l_{12}+l_{23}+\cdots+l_{2p})} \cdots \alpha_p^{r(l_{1p}+l_{2p}+\cdots+l_{p-1p})} \end{aligned} \tag{4.8}$$

where

$$a(k, l_{12}, l_{13}, \ldots, l_{p-1p}) = (-1)^{l_{12}+l_{13}+\cdots+l_{1p}+l_{23}+\cdots+l_{2p}+\cdots+l_{p-1p}} \times \binom{k}{l_{12}}\binom{k}{l_{13}}\cdots\binom{k}{l_{1p}}\binom{k}{l_{23}}\binom{k}{l_{24}}\cdots\binom{k}{l_{2p}}\cdots\binom{k}{l_{p-1p}}. \tag{4.9}$$

Once again, since all integers l_{ij} assume values from zero to k and the coefficients $a(k, l_{12}, l_{13}, \ldots, l_{p-1p})$ are invariant under any permutation of fixed values of these variables, the polynomial (4.8) is equivalent to the S-function series:

$$\sum_{l_{12}l_{13}\ldots l_{1p}}^{k} \sum_{l_{23}l_{24}\ldots l_{2p}}^{k} \cdots \sum_{l_{p-1p}}^{k} a(k, l_{12}, l_{13}, \ldots, l_{p-1p}) \times \{r(l_{12}+l_{13}+\cdots+l_{1p}), r(l_{12}+l_{23}+\cdots+l_{2p}), \ldots, r(l_{1p}+l_{2p}+\cdots+l_{p-1p})\}. \tag{4.10}$$

For example, for $p = 3$, $r = 2$ and $k = 2$ series (4.10) reduces to

$$[A \otimes p_2]^2 = \sum_{l_{12},l_{13},l_{23}} (-1)^{l_{12}+l_{13}+l_{23}} \binom{2}{l_{12}}\binom{2}{l_{13}}\binom{2}{l_{23}} \{2(l_{12}+l_{13}), 2(l_{12}+l_{23}), 2(l_{13}+l_{23})\} \tag{4.11a}$$

or explicitly

$$\begin{aligned}[A \otimes p_2]^2 = {} & \{0\} - 2\{22\} + 2\{211\} + \{44\} - \{431\} + 4\{422\} - 3\{332\} \\ & - 2\{642\} + 2\{633\} + 2\{552\} - 6\{444\} + \{844\} - \{754\} + 4\{664\} - 3\{655\} \\ & - 2\{866\} + 2\{776\} + \{888\}. \end{aligned} \tag{4.11b}$$

4.1.3. S-function series produced by the generating function

$$\prod_{i}^{p} (a_0 + a_1\alpha_i + a_2\alpha_i^2 + a_3\alpha_i^3 + \cdots + a_n\alpha_i^n)^k \qquad \textit{with } k > 0. \tag{4.12}$$

Since

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n)^k = \left(\sum_{q=0}^{n} a_n x^q\right)^k = \sum_{l=0}^{kn} c(a; k, l, n)x^l \tag{4.13}$$

where $c(a; k, l, n)$ are coefficients that incorporate the parameters a_i and give, for particular values of k and n, the occurrence of the term x^l in the expansion, then

$$\begin{aligned}\prod_{i}^{p} (a_0 + a_1\alpha_i + a_2\alpha_i^2 + a_3\alpha_i^3 + \cdots + a_n\alpha_i^n)^k &= \prod_{i}^{p} \sum_{l=0}^{kn} c(a; k, l, n)\alpha_i^l \\ &= \sum_{l_1,l_2,\ldots,l_p=0}^{kn} c(a; k, l_1, n)c(a; k, l_2, n)\ldots c(a; k, l_p, n)\ \alpha_1^{l_1}\alpha_2^{l_2}\ldots\alpha_p^{l_p}. \end{aligned} \tag{4.14}$$

It can be easily checked that expansion (4.14) is a sum of symmetric polynomials, similar to (4.2) and (4.8), thus they correspond to the following S-function series:

$$\sum_{l_1,l_2,\ldots,l_p=0}^{kn} c(a;k,l_1,n)c(a;k,l_2,n)\ldots c(a;k,l_p,n)\ldots\{l_1,l_2,\ldots,l_p\}. \quad (4.15)$$

The multiplicity coefficients $c(a;k,l_i,n)$, obviously dictated from combinatorics, are treated in more detail in section 5.

Some special examples of this series have been studied by Yang and Wybourne (1986), Lascoux and Pragacz (1988) and King *et al* (1989).

Note that the generating function (4.1) can be considered a particular case of the generating function (4.12) with

$$a_i = \begin{cases} 1 & i=0 \\ -1 & i=r \\ 0 & \text{otherwise.} \end{cases}$$

4.2. Infinite series

It is obvious that for negative values of k the generating functions (4.1), (4.6) and (4.12) yield infinite expansions in the indeterminates α_i and therefore generate infinite series of S-functions. Expressing again the generating function as a sum of polynomials of type (3.1), one gets then the expansions

$$\begin{aligned}\prod_i\left(\frac{1}{1-\alpha_i^r}\right)^k &= \prod_i\left(\sum_{l=0}^{\infty}\alpha_i^{rl}\right)^k \\ &= \sum_{l_1,l_2,\ldots,l_p=0}^{\infty} m(k,l_1)m(k,l_2)\cdots m(k,l_p)\ \{rl_1,rl_2,\ldots,rl_p\}\end{aligned} \quad (4.16)$$

$$\begin{aligned}\prod_{i<j}\left(\frac{1}{1-(\alpha_i\alpha_j)^r}\right)^k &= \prod_{i<j}\left(\sum_{l=0}^{\infty}(\alpha_i\alpha_j)^{rl}\right)^k \\ &= \sum_{l_{12},l_{13},\ldots,l_{p-1,p}=0}^{\infty} m(k,l_{12})m(k,l_{13})\cdots m(k,l_{p-1,p}) \\ &\quad\times\{r(l_{12}+l_{13}+\cdots+l_{1p}),r(l_{12}+l_{23}+\cdots+l_{2p}),\cdots,r(l_{1p}+l_{2p}+\cdots+l_{p-1,p})\}\end{aligned} \quad (4.17)$$

$$\begin{aligned}&\prod_i\left(\frac{1}{1+\alpha_i+\alpha_i^2+\cdots+\alpha_i^n}\right)^k \\ &\quad= \sum_{l_1,l_2,\ldots,l_p=0}^{\infty} m(c;k,l_1)m(c;k,l_2)\cdots m(c;k,l_p)\ \{l_1,l_2,\ldots,l_p\}\end{aligned} \quad (4.18)$$

where the multiplicity coefficients $m(k,l)$ and $m(c;k,l)$ are given explicitly in the next section.

5. The multiplicity coefficients

In the actual evaluation of the S-function content of a series one practical aspect to consider is the time it takes to generate all the desired terms. This time can be reduced considerably if an efficient algorithm exists to compute rapidly the multiplicity of each S-function in the series.

A convenient way to implement the calculation of these multiplicities is to identify the coefficients multiplying each S-function, such as $c(a; k, l_i, n)$ or $m(k, l_i)$ in (4.14) and (4.16) respectively, with the elements of a suitable matrix.

Essentially, two types of matrix are needed to evaluate the coefficients appearing in the series presented in section 4. A matrix $M^{(1)}$, from which the coefficients $c(a; k, l, n)$ can be extracted by making the identification

$$c(a; k, l, n) = M^{(1)}(k, l)$$

and whose elements, $M^{(1)}(i, j)$, are given by

$$M^{(1)}(1, j) = \begin{cases} 0 & \text{if } j > n \\ a_j & \text{if } 0 \leqslant j \leqslant n \end{cases} \tag{5.1a}$$

$$M^{(1)}(i, j) = \begin{cases} 0 & \text{for } j > in \\ \displaystyle\sum_{\nu=0}^{j} M^{(1)}(i-1, \nu) \times M^{(1)}(1, j-\nu) & \text{for } 0 \leqslant j \leqslant in \end{cases} \tag{5.1b}$$

and a matrix $M^{(2)}$, with an infinite number of columns, whose elements, defined by

$$M^{(2)}(i, j) = \begin{cases} a_j & \text{for } i = 1 \text{ and } j = 0, 1, \ldots, \infty \\ \displaystyle\sum_{\nu=0}^{j} M^{(2)}(i-1, \nu) \times M^{(2)}(1, j-\nu) & \text{for } i \geqslant 2 \text{ and } j = 0, 1, \ldots, \infty \end{cases} \tag{5.2}$$

give the coefficients $m(k, l)$ through the relation

$$m(k, l) = M^{(2)}(k, l).$$

The set up of the above matrices follows straightforwardly from a generalization of the binomial formula.

For example, the coefficients required in the expansion of series (4.12) with $n = 4$, $k = 3$ and $a_0 = a_1 = a_2 = a_3 = a_4 = 1$ are obtained from the third row of the matrix

$$M^{(1)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 6 & 10 & 15 & 18 & 19 & 18 & 15 & 10 & 6 & 3 & 1 \end{pmatrix}. \tag{5.3}$$

On the other hand, infinite expansions such as

$$\left(\frac{1}{1-x^r}\right)^k = \left(\sum_{\mu=0}^{\infty} x^{r\mu}\right)^k = \sum_{l=0}^{\infty} m(k, l) x^{rl} \tag{5.4}$$

$$\left(\frac{1}{1+x^r}\right)^k = \left(\sum_{\mu=0}^{\infty} (-1)^{\mu} x^{r\mu}\right)^k = \sum_{l=0}^{\infty} m(k, l) x^{rl} \tag{5.5}$$

$$\left(\frac{1}{1-\beta x^r}\right)^k = \left(\sum_{\mu=0}^{\infty} \beta^{\mu} x^{r\mu}\right)^k = \sum_{l=0}^{\infty} m(k, l) x^{rl} \tag{5.6}$$

involve the coefficients $m(k, l)$ which can be calculated from a matrix $M^{(2)}$ with the elements of the first row being

$$M^{(2)}(1,l) = \begin{cases} 1 & \text{for (5.4)} \\ (-1)^l & \text{for (5.5)} \\ \beta^l & \text{for (5.6).} \end{cases}$$

So, the coefficients needed in expansions (4.16) or (4.17) when, for example, $k = 4$ are given by the fourth row of the matrix

$$M^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\ 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & 55 & 66 & 78 & \cdots \\ 1 & 4 & 10 & 20 & 35 & 56 & 84 & 120 & 165 & 220 & 286 & 364 & \cdots \end{pmatrix}. \tag{5.7}$$

Now, the coefficients $m(c; k, l)$, though not simple coefficients like the other two, can also be quickly computed. The coefficient $m(c; k, l)$ could be called a 'compound coefficient' since, as shown in the following analysis, one needs both types of matrix to determine it. In fact,

$$\begin{aligned} \left(\frac{1}{(1+x+x^2+\cdots+x^n)}\right)^k &= \left(\sum_{\xi=0}^{\infty}(-1)^{\xi}(x+x^2+\cdots+x^n)^{\xi}\right)^k \\ &= \left(1+\sum_{\xi=1}^{\infty}(-1)^{\xi}\sum_{\mu=1}^{\xi n}c(a;\xi,\mu,n)x^{\mu}\right)^k = \left(1+\sum_{j=1}^{\infty}c_j x^j\right)^k = \sum_{l=0}^{\infty}m(c;k,l)x^l \end{aligned} \tag{5.8}$$

where the coefficients $c(a; \xi, \mu, n)$ are first obtained from the ξth row of a matrix $M^{(1)}$ with parameters $a_0 = 0$, $a_1 = a_2 = \cdots = a_n = 1$ and then the coefficients $m(c; k, l)$ are extracted from the kth row of a matrix of type $M^{(2)}$ whose elements on the first row are

$$M^{(2)}(1,j) = \begin{cases} 1 & \text{for } j = 0 \\ c_j = \sum_{\xi=1}^{j}(-1)^{\xi}c(a;\xi,j,n) & \text{otherwise.} \end{cases}$$

6. Aplications

6.1. Generating other series

From the three basic types (finite and infinite) of series given in section 4, others, more or less elaborate, can be derived easily. Take, for example, the so-called series C (Black *et al* 1983) whose generating function is

$$\prod_{i\leqslant j}(1-\alpha_i\alpha_j) = \prod_i(1-\alpha_i^2)\prod_{j<k}(1-\alpha_j\alpha_k). \tag{6.1}$$

This function is obviously the product of generating functions (4.1) (with $r=2$ and $k=1$) and (4.6) (with $r=1$ and $k=1$). Expanding (6.1)

$$C = \sum_{m_1,m_2,\ldots,m_p=0}^{1} (-1)^{m_1+m_2+\cdots+m_p} \alpha_1^{2m_1}\alpha_2^{2m_2}\cdots\alpha_p^{2m_p}$$

$$\times \sum_{l_{12},l_{13},\ldots,l_{p-1,p}=0}^{1} (-1)^{l_{12}+l_{13}+\cdots+l_{p-1,p}} \alpha_1^{l_{12}+\cdots+l_{1p}} \alpha_2^{l_{12}+\cdots+l_{2p}} \cdots \alpha_p^{l_{1p}+\cdots+l_{p-1,p}}$$

$$= \sum_{m_1,m_2,\ldots,m_p} \sum_{l_{12},l_{13},\ldots,l_{p-1,p}} (-1)^{m_1+m_2+\cdots+m_p}(-1)^{l_{12}+l_{13}+\cdots+l_{p-1,p}}$$

$$\times \sum \alpha_1^{2m_1+l_{12}+\cdots+l_{1p}} \alpha_2^{2m_2+l_{12}+\cdots+l_{2p}} \cdots \alpha_p^{2m_p+l_{1p}+\cdots+l_{p-1,p}} \qquad (6.2a)$$

where again

$$\sum \alpha_1^{2m_1+l_{12}+\cdots+l_{1p}} \alpha_2^{2m_2+l_{12}+\cdots+l_{2p}} \cdots \alpha_p^{2m_p+l_{1p}+\cdots+l_{p-1,p}}$$

corresponds (cf (3.4)) to $p!$ S-functions of fixed values m_i and l_{ij}. Thus,

$$C = \sum_{m_1,m_2,\ldots,m_p=0}^{1} \sum_{l_{12},l_{13},\ldots,l_{p-1,p}=0}^{1} (-1)^{m_1+m_2+\cdots+m_p}(-1)^{l_{12}+l_{13}+\cdots+l_{p-1,p}}$$

$$\times \{2m_1+l_{12}+\cdots+l_{1p}, 2m_2+l_{12}+\cdots+l_{2p}, \cdots, 2m_p+l_{1p}+\cdots+l_{p-1,p}\}. \qquad (6.2b)$$

Comparing result (6.2b) with (4.4) and (4.10) one can then generalize and give the two following rules (cf also appendix A):

When a compound generating function is made of partial generating functions whose S-function content is known, then

(i) each S-function, generated by the compound function, is derived from S-functions in the partial expansions (one from each), by adding their corresponding parts;

(ii) the coefficient associated with each new S-function is the product of the coefficients, in the partial expansions, associated with the S-functions that give rise to this particular new S-function.

If one restricts (6.2b) to S-functions with no more than two parts ($p=2$) series C reduces to

$$C = \sum_{m_1,m_2=0}^{1} \sum_{l=0}^{1} (-1)^{m_1+m_2+l}\{2m_1+l, 2m_2+l\}. \qquad (6.3)$$

The possible combinations of values assumed by m_1, m_2 and l and the S-functions to which they give rise are summarized in table 1.

Since $-\{02\} = +\{11\}$ and $+\{13\} = -\{22\}$ series C reduces further to

$$C = \{0\} - \{2\} + \{31\} - \{33\}. \qquad (6.4)$$

Table 1. Terms in the expansion of (6.3).

m_1	m_2	l	$\{\lambda_1\lambda_2\}$
0	0	0	$+\{0\}$
1	0	0	$-\{2\}$
0	1	0	$-\{02\}$
0	0	1	$-\{11\}$
1	1	0	$+\{22\}$
1	0	1	$+\{31\}$
0	1	1	$+\{13\}$
1	1	1	$-\{33\}$

The series $C^3 = C \times C \times C$ is equally easy to generate. Restricting again to S-functions divided into no more than two parts, this new series is, in compact form,

$$C^3 = \sum_{m_1,m_2=0}^{3} \sum_{l=0}^{3} (-1)^{m_1+m_2+l} \binom{3}{m_1} \binom{3}{m_2} \binom{3}{l} \{2m_1 + l, 2m_2 + l\}. \tag{6.5}$$

The set of S-functions that result from all possible combinations of values assumed by m_1, m_2 and l (from zero to three) are, after reduction of equivalent S-functions,

$$\begin{aligned} C^3 = \{0\} &- 3\{2\} + 3\{22\} + 6\{31\} + 3\{4\} - 10\{33\} \\ &- 9\{42\} - 8\{51\} - \{6\} + 6\{44\} + 18\{53\} + 9\{62\} + 3\{71\} \\ &- 6\{55\} - 18\{64\} - 9\{73\} - 3\{82\} + 10\{66\} + 9\{75\} + 8\{84\} \\ &+ \{93\} - 3\{77\} - 6\{86\} - 3\{95\} + 3\{97\} - \{99\}. \end{aligned} \tag{6.6}$$

Clearly, one advantage of this method is the possibility of deriving the S-function content of the product of k copies of a given series (or the product of any number of series) without having to determine the explicit expansion of each constituent series first.

6.2. *Occurrence of a particular S-function in a series*

The method advocated in this paper is particularly appropriate to the calculation of the multiplicity of a given standard S-function in a series without having to generate the whole series. Consider, for example, the multiplicity of the S-function $\{444\}$ in the series D^n, $n \geqslant 1$, generated by

$$\prod_{i \leqslant j} \left(\frac{1}{1 - \alpha_i \alpha_j} \right)^n = \prod_i \left(\frac{1}{1 - \alpha_i^2} \right)^n \prod_{k<l} \left(\frac{1}{1 - \alpha_k \alpha_l} \right)^n. \tag{6.7}$$

Since $\{444\}$ is equivalent to the non-standard S-functions $\{354\}$, $\{435\}$, $\{336\}$, $\{255\}$ and $\{246\}$ (cf 2.9), their multiplicities in expansion (6.7) have also to be taken into account.

The S-function content of (6.7) is, according to (4.16) and (4.17),

$$\sum_{\nu_1,\nu_2,\nu_3=0}^{\infty} \sum_{l_1,l_2,l_3=0}^{\infty} c(n, \nu_1, \nu_2, \nu_3, l_1, l_2, l_3) \times \{2\nu_1 + l_1 + l_3, 2\nu_2 + l_1 + l_2, 2\nu_3 + l_2 + l_3\} \tag{6.8}$$

with

$$c(n, \nu_1, \nu_2, \nu_3, l_1, l_2, l_3) = m(n, \nu_1)m(n, \nu_2)m(n, \nu_3)m(n, l_1)m(n, l_2)m(n, l_3)$$

where the restriction to $p = 3$ is justified since S-function {444} and its non-standard equivalents have length three. The coefficients $m(n, \nu_i)$ and $m(n, l_i)$, $i = 1, 2, 3$, are obtained from the nth row of matrix (5.7).

Table 2 shows the multiplicities of the above mentioned S-functions in the series D^n for $n = 3$ and $n = 10$. These multiplicitites are obtained by adding up the coefficients $c(n, \nu_1, \nu_2, \nu_3, l_1, l_2, l_3)$ corresponding to all possible combinations of ν_i and l_i that give rise to the particular S-function. The possible combinations of ν_i and l_i and their associated coefficients are given in tables 1–6 of appendix B.

Table 2. Multiplicities of {444}, {354}, {435}, {246}, {255} and {336} in expansion (6.8) for $n = 3$ and $n = 10$.

S-function	Mult. in D^3	Mult. in D^{10}
{444}	4185	3 925 725
{354} = −{444}	3555	3 333 00
{435} = −{444}	3555	3 333 000
{246} = −{444}	2421	2 060 575
{255} = +{444}	2610	213 520
{336} = +{444}	2792	2 516 800

One can then conclude that the total multiplicity of {444} is 56 in the series D^3 and 29470 in D^{10}. It should be emphasized the fact that with this method both numbers are equally easy and fast to calculate.

6.3. *Outer product and skew division of an S-function by a series*

Another attractive feature of this method is the fact that one can easily evaluate the outer product or skew division of an arbitrary S-function by any series, without having to derive, beforehand, the series explicitly.

Consider the outer product of the S-function $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ by the series whose generating function is

$$\prod_i (1 - 2\alpha_i + \alpha_i^2 - 2\alpha_i^3)^2 = \sum_{l_1, l_2, \ldots, l_p} c(a; 2, l_1, 3)c(a; 2, l_2, 3)\ldots c(a; 2, l_p, 3)\alpha_1^{l_1}\alpha_2^{l_2}\ldots\alpha_p^{l_p}. \tag{6.9}$$

The variables $l_1, l_2, \ldots, l_p$ can take any integer value from zero to six and the corresponding coefficients are given by the second row of a $M^{(1)}$ type matrix

$$\begin{pmatrix} 1 & -2 & 1 & -2 & 0 & 0 & 0 \\ 1 & -4 & 6 & -8 & 9 & -4 & 4 \end{pmatrix}. \tag{6.10}$$

Expressing $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ in determinantal form

$$\begin{pmatrix} \alpha_1^{\lambda_1+p-1} & \alpha_1^{\lambda_2+p-2} & \cdots & \alpha_1^{\lambda_p} \\ \alpha_2^{\lambda_1+p-1} & \alpha_2^{\lambda_2+p-2} & \cdots & \alpha_2^{\lambda_p} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_p^{\lambda_1+p-1} & \alpha_p^{\lambda_2+p-2} & \cdots & \alpha_p^{\lambda_p} \end{pmatrix} / \Delta(\alpha) \tag{6.11}$$

it is immediate that multiplication by the polynomial function (6.9) yields the series

$$\sum_{l_1,l_2,\ldots,l_p} c(a;2,l_1,3)\ldots c(a;2,l_p,3)\{\lambda_1+l_1,\lambda_2+l_2,\ldots,\lambda_p+l_p\}. \qquad (6.12)$$

On the other hand, the skew division of $\{\lambda_1,\lambda_2,\ldots,\lambda_p\}$ by the same series is obtained by multiplying the determinant (6.11) by

$$\sum_{l_1,l_2,\ldots,l_p} c(a;2,l_1,3)c(a;2,l_2,3)\ldots c(a;2,l_p,3)\alpha_1^{-l_1}\alpha_2^{-l_2}\ldots\alpha_p^{-l_p}. \qquad (6.13)$$

The result is then

$$\sum_{l_1,l_2,\ldots,l_p} c(a;2,l_1,3)c(a;2,l_2,3)\ldots c(a;2,l_p,3)\{\lambda_1-l_1,\lambda_2-l_2,\ldots,\lambda_p-l_p\}. \qquad (6.14)$$

So, for example, the skew division of {222} by the above series is in compact form

$$\sum_{l_1,l_2,l_3} c(a;2,l_1,3)c(a;2,l_2,3)c(a;2,l_3,3)\ \{2-l_1,2-l_2,2-l_3\}. \qquad (6.15a)$$

Note that in a skew division the resulting series can never contain S-functions with a length greater than the length of the S-function being divided. So (6.15*a*) has naturally been restricted to three parts. The series (6.15*a*) can easily be expanded to give

$$\{222\}-4\{221\}+6\{22\}+10\{211\}-16\{21\}-24\{111\}+4\{2\}+37\{11\}-4\{1\}-14\{0\}. \qquad (6.15b)$$

As an example of the outer product let us consider the S-function to be {2} and the series the same as above, (6.9). The result is

$$\sum_{l_1,l_2,\ldots,l_p} c(a;2,l_1,3)c(a;2,l_2,3)\ldots c(a;2,l_p,3)\{2+l_1,l_2,\ldots,l_p\}. \qquad (6.16a)$$

The maximum number of parts that the S-functions in the resulting series are allowed to have, has to be chosen now in order to truncate it. Taking $p=2$ so that the list does not become tediously long, one has the following series

$$\begin{aligned}\{22\}&-4\{32\}+6\{42\}+10\{33\}-8\{52\}-16\{43\}+9\{62\}\\ &+23\{53\}+4\{44\}-4\{72\}-32\{63\}-12\{54\}+4\{82\}+12\{73\}\\ &+38\{64\}+10\{55\}-16\{83\}-8\{74\}-48\{65\}+24\{84\}+8\{75\}+49\{66\}\\ &-32\{85\}-4\{76\}+36\{86\}-20\{77\}-16\{87\}+16\{88\}. \qquad (6.17)\end{aligned}$$

In conclusion, obtaining a unified procedure for the actual evaluation of the S-function content of any homogeneous polynomial function whether it be the generating function of a known classical series or any other function not yet explored was attempted. Although the results of the examples given here can be checked by hand they have been generated by computer codes able to handle more elaborated situations. The method is such that programs can be straightforwardly customized.

Appendix A

Consider the two series whose generating functions are

$$\prod_i \left(\frac{1}{1-\alpha_i^2}\right)^2 \tag{A.1}$$

and its inverse

$$\prod_i (1-\alpha_i^2)^2. \tag{A.2}$$

The result of the outer product of these two series is obviously $\{0\} = 1$. Let us see that this simple result follows from the rules, given in section 6, for determining the S-function content of a compound generating function from the S-function content of the composing functions.

Given that the S-function content of (A.1) is

$$\sum_{l_1,l_2,\ldots,l_p=0}^{\infty} m(2,l_1)m(2,l_2)\ldots m(2,l_p)\{2l_1,2l_2,\ldots,2l_p\} \tag{A.3}$$

and of (A.2)

$$\sum_{m_1,m_2,\ldots,m_p=0}^{2} \binom{2}{m_1}\binom{2}{m_2}\cdots\binom{2}{m_p}(-1)^{m_1+m_2+\cdots+m_p}\{2m_1,2m_2,\ldots,2m_p\} \tag{A.4}$$

then the S-function content of the generating function which is the product of (A.1) and (A.2) is

$$\sum_{l_1,l_2,\ldots,l_p=0}^{\infty}\ \sum_{m_1,m_2,\ldots,m_p=0}^{2} m(2,l_1)m(2,l_2)\ldots m(2,l_p)\binom{2}{m_1}\binom{2}{m_2}\cdots\binom{2}{m_p}$$
$$\times(-1)^{m_1+m_2+\cdots+m_p}\{2l_1+2m_1,2l_2+2m_2,\ldots,2l_p+2m_p\}. \tag{A.5}$$

In order to prove that expansion (A.5) reduces to a single term, $\{0\}$, note that any part i, $\lambda_i = 2l_i + 2m_i$, of the S-functions in (A.5) can only assume even integer values and that an arbitrary even integer, say $2k$, can be obtained in three ways:

$$\begin{cases} m_i = k, l_i = 0 & \text{with a coefficient} = k+1 \\ m_i = k-1, l_i = 1 & \text{with a coefficient} = -2(k) \\ m_i = k-2, l_i = 2 & \text{with a coefficient} = k-1 \end{cases}$$

where the associated coefficients are immediately derived from

$$m(2,l_i) = l_i + 1$$

and

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1.$$

Since the different parts assume values independently of each other then, with the exception of $\{0\}$, all other S-functions appear with multiplicity zero.

Appendix B

Table A1. Combinations of ν_i and l_i yielding {444}. The coefficient associated with each combination is given in the last two columns for series D^3 and D^{10} respectively. The multiplicity of {444} in D^3 is 4185 and in D^{10} 3 925 725.

ν_1	ν_2	ν_3	l_1	l_2	l_3	Coeff.	Coeff.
0	0	0	2	2	2	216	166 375
0	0	1	3	1	1	270	220 000
0	0	2	4	0	0	90	39 325
0	1	0	1	1	3	270	220 000
0	1	1	2	0	2	324	302 500
0	2	0	0	0	4	90	39 325
1	0	0	1	3	1	270	220 000
1	0	1	2	2	0	324	302 500
1	1	0	0	2	2	324	302 500
1	1	1	1	1	1	729	1 000 000
1	1	2	2	0	0	324	302 500
1	2	1	0	0	2	324	302 500
2	0	0	0	4	0	90	39 325
2	1	1	0	2	0	324	302 500
2	2	2	0	0	0	216	166 375

Table A2. Combinations of ν_i and l_i yielding {354}. The coefficient associated with each combination is given in the last two columns for series D^3 and D^{10} respectively. The multiplicity of {354} in D^3 is 3555 and in D^{10} 3 333 000.

ν_1	ν_2	ν_3	l_1	l_2	l_3	Coeff.	Coeff.
0	0	0	2	3	1	180	121 000
0	0	1	3	2	0	180	121 000
0	1	0	1	2	2	324	302 500
0	1	1	2	1	1	486	550 000
0	1	2	3	0	0	180	121 000
0	2	0	0	1	3	180	121 000
0	2	1	1	0	2	324	302 500
1	0	0	1	4	0	135	71 500
1	1	0	0	3	1	270	220 000
1	1	1	1	2	0	486	550 000
1	2	1	0	1	1	486	550 000
1	2	2	1	0	0	324	302 500

Table A3. Combinations of ν_i and l_i yielding {435}. The coefficient associated with each combination is given in the last two columns for series D^3 and D^{10} respectively. The multiplicity of {435} in D^3 is 3555 and in D^{10} 3 333 000.

ν_1	ν_2	ν_3	l_1	l_2	l_3	Coeff.	Coeff.
0	0	0	1	2	3	180	121 000
0	0	1	2	1	2	324	302 500
0	0	2	3	0	1	180	121 000
0	1	0	0	1	4	135	71 500
0	1	1	1	0	3	270	220 000
1	0	0	0	3	2	180	121 000
1	0	1	1	2	1	486	550 000
1	0	2	2	1	0	324	302 500
1	1	1	0	1	2	486	550 000
1	1	2	1	0	1	486	550 000
2	0	1	0	3	0	180	121 000
2	1	2	0	1	0	324	302 500

Table A4. Combinations of ν_i and l_i yielding {246}. The coefficient associated with each combination is given in the last two columns for series D^3 and D^{10} respectively. The multiplicity of {246} in D^3 is 2421 and in D^{10} 2 060 575.

ν_1	ν_2	ν_3	l_1	l_2	l_3	Coeff.	Coeff.
0	0	0	0	4	2	90	39 325
0	0	1	1	3	1	270	220 000
0	0	2	2	2	0	216	166 375
0	1	1	0	2	2	324	302 500
0	1	2	1	1	1	486	550 000
0	1	3	2	0	0	180	121 000
0	2	2	0	0	2	216	166 375
1	0	1	0	4	0	135	71 500
1	1	2	0	2	0	324	302 500
1	2	3	0	0	0	180	121 000

Table A5. Combinations of ν_i and l_i yielding {255}. The coefficient associated with each combination is given in the last two columns for series D^3 and D^{10} respectively. The multiplicity of {255} in D^3 is 2610 and in D^{10} 2 313 520.

ν_1	ν_2	ν_3	l_1	l_2	l_3	Coeff.	Coeff.
0	0	0	1	4	1	135	71 500
0	0	1	2	3	0	180	121 000
0	1	0	0	3	2	180	121 000
0	1	1	1	2	1	486	550 000
0	1	2	2	1	0	324	302 500
0	2	1	0	1	2	324	302 500
0	2	2	1	0	1	324	302 500
1	0	0	0	5	0	63	20 020
1	1	1	0	3	0	270	220 000
1	2	2	0	1	0	324	302 500

Table A6. Combinations of ν_i and l_i yielding {336}. The coefficient associated with each combination is given in the last two columns for series D^3 and D^{10} respectively. The multiplicity of {336} in D^3 is 2792 and in D^{10} 2 516 800.

ν_1	ν_2	ν_3	l_1	l_2	l_3	Coeff.	Coeff.
0	0	0	0	3	3	100	48 400
0	0	1	1	2	2	324	302 500
0	0	2	2	1	1	324	302 500
0	0	3	3	0	0	100	48 400
0	1	1	0	1	3	270	220 000
0	1	2	1	0	2	324	302 500
1	0	1	0	3	1	270	220 000
1	0	2	1	2	0	324	302 500
1	1	2	0	1	1	486	550 000
1	1	3	1	0	0	270	220 000

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